Computation Algebraic Topology Mini-Project - Discrete Morse Theory for Relative Cosheaf Homology

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Abstract

In the context of computationally studying topological spaces, many algebraic tools have proven effective both theoretically and practically. Specifically, given a filtration of simplicial complexes, one can compute relative and persistence homology to determine properties of the topological object at hand. Moreover, one can attach additional data to the complex using (co)sheaves and adapt the relevant algebraic tools accordingly. However, computations can involve linear algebra on huge spaces, dimensions of which are linked to the number of manipulated simplicies. Discrete Morse theory has been developed to tackle this issue of huge practical importance, as it is able to drastically reduce the effective dimension of the linear spaces by removing most of the simplicies, while computing the same algebraic objects (such as homology groups and their corresponding maps). In this essay, we will work toward a generalization of discrete Morse theory in the context of cosheaves. The ultimate goal is to describe how to construct relative cosheaf homology on filtrations of finite simplicial complexes, adapting discrete Morse theory to this context. In particular we will underline how this simplifies computation of the associated long exact sequence. As a final generalization, we will work on longer filtrations and apply discrete Morse theory to persistent cosheaf homology.

Contents

1 Introduction

The theorems we will mention without proof are all properly introduced in the lecture [Nan]. This work naturally builds on top of all the material introduced there. To make the reader's life easier, we have carefully made sure to use the same notations. Adapting discrete Morse theory to the context of cosheaves has been largely inspired by the paper [CGN15], but we take a somewhat different route, being closer to the spirit of the lecture notes.

After a brief introduction of cosheaves over finite simplicial complexes, we will focus on relative homology in Section 2. As relative homology works on a filtration of a simplicial complex, we won't explicitly talk about persistence until the final section, as the work is very similar. Once relative homology will be adapted to the context of cosheaves, we will work toward building discrete Morse theory for cosheaves in Section 3, thus providing a recipe to drastically improve computations. In Section 4 we will implement discrete Morse theory to relative cosheaf homology, and explicitly describe how the computation of maps of the associated long exact sequence can be sped up. Finally we will do the same with longer filtrations in Section 5 with persistent cosheaf homology.

2 Cosheaves over Finite Simplicial Complexes

For the sake of the following presentation, define a filtration of three simplicial complexes $M \subset L \subset K$.

2.1 Cosheaf

Let's introduce the underlying object of all our subsequent work.

Definition 1. *A **cosheaf** over a simplicial complex K is a covariant functor $\mathcal{C} : (K, \geq) \to$ **Vect**$_{\mathbb{F}}$. The vector spaces denoted by $\mathcal{C}(\tau)$ are called costalks the linear maps $\mathcal{C}(\tau' \geq \tau)$ between them are called exclusion maps.*

Please refer to [Nan] for a more thorough presentation of such an object. Its useful properties we don't describe further have to do with its functoriality. Here are some straightforward example of cosheaves on a simplicial complex K:

- The zero cosheaf 0_K which assigns the trivial vector space to every simplex. All exclusion maps $\mathcal{C}(\tau' \geq \tau)$ are zero.
- Given a simplex $\tau \in K$, the skyscraper cosheaf $\mathcal{C}_\tau$. It assigns the trivial vector space to every simplex except τ, for which $\mathcal{C}(\tau) = \mathbb{F}$. All the exclusion maps are zero except $\mathcal{C}(\tau \geq \tau) = Id$.
- The constant cosheaf $\mathbb{F}_K$, which assigns the one-dimensional costalk to every simplex of K and the identity exclusion map to every face relations. We will see that this simple cosheaf recovers the usual formulation of our standard algebraic objects (chain complexes, homology groups etc.), which are thus special cases of this encompassing theory.

Now, we naturally reformulate the usual algebraic objects in the theory of computational algebraic topology, as introduced in [Nan], to accommodate for cosheaves. In particular, every cosheaf $\mathcal{C}$ on K induces a chain complex

$$\cdots \longrightarrow \boldsymbol{C}_2(K;\mathcal{C}) \xrightarrow{\partial_2^{\mathcal{C}}} \boldsymbol{C}_1(K;\mathcal{C}) \xrightarrow{\partial_1^{\mathcal{C}}} \boldsymbol{C}_0(K;\mathcal{C}) \xrightarrow{\partial_0^{\mathcal{C}}} 0$$

which gives rise to the homology of K with coefficients in $\mathcal{C}$.

Definition 2. *For each dimension $k \geq 0$, the vector space of k-chains of K with $\mathcal{C}$- coefficients is the product*

$$\boldsymbol{C}_k(K;\mathcal{C}) = \prod_{dim(\tau)=k} \mathcal{C}(\tau)$$

of the costalks of $\mathcal{C}$ over all the k-dimensional simplicies of K.

Assume that the simplicies of K are ordered such that each simplex τ of dimension greater than 1 has an i-th face τ_{-i}. Write

$$[\sigma : \tau] = \begin{cases} +1 & \text{if } \sigma = \tau_{-i},\ i \text{ even} \\ -1 & \text{if } \sigma = \tau_{-i},\ i \text{ odd} \\ 0 & \text{else} \end{cases}$$

so that $[\sigma : \tau]$ is the coefficient of σ in the simplicial boundary of τ. This lets us define the boundary operator in the context of cosheaves.

Definition 3. *For each $k \geq 0$, the k-th boundary map of K with $\mathcal{C}$-coefficients is the linear map*

$$\partial_k^{\mathcal{C}} = \boldsymbol{C}_k(K,\mathcal{C}) \longrightarrow \boldsymbol{C}_{k-1}(K,\mathcal{C})$$

Defined via the following block-action: for each pair of simplicies $\tau > \sigma$ with $dim(\tau) = dim(\sigma) + 1 = k$, the $\mathcal{C}(\tau) \longrightarrow \mathcal{C}(\sigma)$ component of $\partial_k^{\mathcal{C}}$ is equal to

$$\partial_k^{\mathcal{C}}\big|_{\sigma,\tau} = [\sigma : \tau]\, \mathcal{C}(\tau \geq \sigma).$$

Block actions are zero elsewhere.

As promised, we now prove that we have constructed a well-defined chain complex.

Theorem 2.1. *$(\boldsymbol{C}_\bullet(K;\mathcal{C}), \partial_\bullet^{\mathcal{C}})$ forms a chain complex. In other words, $\partial_{k-1}^{\mathcal{C}} \circ \partial_k^{\mathcal{C}} = 0$.*

Proof. It suffices to prove that for each k-simplex τ'' and (k-2) simplex τ the $\mathcal{C}(\tau'') \to \mathcal{C}(\tau)$ block of this composite is the zero map. We compute

$$\begin{aligned}
\partial_{k-1}^{\mathcal{C}} \circ \partial_k^{\mathcal{C}}\big|_{\tau,\tau''} &= \sum_{\dim(\tau')=k} \partial_{k-1}^{\mathcal{C}}\big|_{\tau,\tau'} \circ \partial_k^{\mathcal{C}}\big|_{\tau',\tau''} \\
&= \sum_{\tau<\tau'<\tau''} \partial_{k-1}^{\mathcal{C}}\big|_{\tau,\tau'} \circ \partial_k^{\mathcal{C}}\big|_{\tau',\tau''} \\
&= \sum_{\tau<\tau'<\tau''} [\tau:\tau']\,[\tau':\tau'']\,\mathcal{C}(\tau' \geq \tau) \circ \mathcal{C}(\tau'' \geq \tau') \\
&= \sum_{\tau<\tau'<\tau''} [\tau:\tau']\,[\tau':\tau'']\,\mathcal{C}(\tau'' \geq \tau) \quad \text{by functoriality (associativity)} \\
&= \left(\sum_{\tau<\tau'<\tau''} [\tau:\tau']\,[\tau':\tau''] \right) \mathcal{C}(\tau'' \geq \tau)
\end{aligned}$$

But $\left(\sum_{\tau<\tau'<\tau''} [\tau:\tau']\,[\tau':\tau'']\right) = 0$ because it is the coefficient of τ in the composite $\partial_{k-1}^K \circ \partial_k^K(\tau'')$, which is null, of course. This concludes the proof. ☐

It is therefore straightforward to define homology groups in the context of cosheaves.

Definition 4. *For each dimension $k \geq 0$, the k-th homology group of K with coefficient in $\mathcal{C}$ is the quotient vector space*

$$\boldsymbol{H}_k(K;\mathcal{C}) = \frac{\textit{ker}\ \partial_k^{\mathcal{C}}}{\textit{img}\ \partial_{k+1}^{\mathcal{C}}}$$

It is all the more quite obvious how this cosheaf homology agrees with standard homology whenever $\mathcal{C}$ is the constant cosheaf $\mathbb{F}_K$, since $\boldsymbol{C}(K;\mathbb{F}_K) = \boldsymbol{C}(K;\mathbb{F})$ and there is an equality of boundary operators $\partial_\bullet^{\mathbb{F}_K} = \partial_\bullet^K$, as can be inferred from their respective definition.

2.2 Relative Cosheaf Homology

We will now direct our work so as to adapt relative homology to the context of cosheaves. Again, in this regard, consider a filtration $M \subset L \subset K$. For ease of notation, omit the superscript $\mathcal{C}$ in the boundary operator, which can be inferred implicitly anyway.

Task 1. Remember that we have $\boldsymbol{C}_k(K;\mathcal{C}) = \prod_{\dim(\tau)=k} \mathcal{C}(\tau)$. Thus $\boldsymbol{C}_k(L;\mathcal{C})$ is a normal subgroup of $\boldsymbol{C}_k(K;\mathcal{C})$. The k-chains of K relative to L with coefficient in $\mathcal{C}$ is then well-defined object

$$\boldsymbol{C}_k(K,L;\mathcal{C}) = \frac{\boldsymbol{C}_k(K;\mathcal{C})}{\boldsymbol{C}_k(L;\mathcal{C})}$$

Moreover, the boundary operator on K is such that its restriction to L is the boundary operator on L:

$$\partial_\bullet^K\big|_L = \partial_\bullet^L$$

This induces a well defined boundary operator $\partial_\bullet^{K,M}$ on the chain $\boldsymbol{C}_\bullet(K,L;\mathcal{C})$. Remark all the more that $\partial^{K,M} \circ \partial^{K,M} = 0$. This proves the following;

Theorem 2.2. *$(\boldsymbol{C}_\bullet(K,L;\mathcal{C}), \partial_\bullet^{K,L})$ forms a chain complex.*

Finally, we are able to define the associated relative cosheaf homology groups.

Definition 5. *The relative cosheaf homology groups $\boldsymbol{H}_k(K,L;\mathcal{C})$ are defined to be the homology groups of the chain complex $(\boldsymbol{C}_\bullet(K,L;\mathcal{C}), \partial_\bullet^{K,L})$.*

Remark When $\mathcal{C}$ is the constant cosheaf $\mathbb{F}_K$, all the different objects, chain complexes and boundary operators, fall back to their regular definition. Thus the relative cosheaf homology groups $\boldsymbol{H}_k(K, L; \mathbb{F}_K)$ also become the standard cosheaf homology groups $\boldsymbol{H}_k(K, L)$.

Task 2. The obvious inclusion map $\boldsymbol{C}_\bullet(L;\mathcal{C}) \to \boldsymbol{C}_\bullet(K;\mathcal{C})$ induces, after taking the quotient by $\boldsymbol{C}_\bullet(M;\mathcal{C})$, the resulting inclusion map:

$$p_\bullet : \boldsymbol{C}_\bullet(L, M;\mathcal{C}) \to \boldsymbol{C}_\bullet(K, M;\mathcal{C})$$

From basic algebra theory we know that

$$\frac{\boldsymbol{C}_\bullet(K, M;\mathcal{C})}{\boldsymbol{C}_\bullet(L, M;\mathcal{C})} \sim \boldsymbol{C}_\bullet(K, L;\mathcal{C}).$$

From this we naturally define the projection map

$$q_\bullet : \boldsymbol{C}_\bullet(K, M;\mathcal{C}) \longrightarrow \boldsymbol{C}_\bullet(K, L;\mathcal{C}).$$

And of course since $\ker(q_\bullet) = \operatorname{im}(p_\bullet) = \boldsymbol{C}_\bullet(L, M;\mathcal{C})$, the following short sequence is exact:

$$0 \longrightarrow \boldsymbol{C}_\bullet(L, M;\mathcal{C}) \xrightarrow{p_\bullet} \boldsymbol{C}_\bullet(K, M;\mathcal{C}) \xrightarrow{q_\bullet} \boldsymbol{C}_\bullet(K, L;\mathcal{C}) \longrightarrow 0$$

We now prove that these maps are indeed chain maps (i.e they commute with the boundary operator).

Theorem 2.3. *$p_\bullet$ is a chain map.*

Proof. Take $\tau \in \boldsymbol{C}_k(L;\mathcal{C})$. For convenience we will now omit the subscripts in our maps, as they can be recovered implicitly anyway. Define $\pi_M : \boldsymbol{C}_\bullet(L;\mathcal{C}) \to \boldsymbol{C}_\bullet(L, M;\mathcal{C})$ as the projection in the quotient space. Then:

$$\begin{aligned}
\partial^{K,M} \circ p \circ \pi_M(\tau) &= \partial^{K,M} \circ \pi_M(\tau) \\
&= \pi_M \circ \partial^K(\tau) && \text{by definition of the boundary on quotient space} \\
&= \pi_M \circ \partial^L(\tau) && \text{because } \partial^K\big|_L = \partial^L \\
&= p \circ \pi_M \circ \partial^L(\tau) \\
&= p \circ \partial^{L,M} \circ \pi_M(\tau)
\end{aligned}$$

Thus $\partial^{K,M} \circ p = p \circ \partial^{L,M}$ and p is a chain map. □

Theorem 2.4. *$q_\bullet$ is a chain map.*

Proof. Take $\tau \in \boldsymbol{C}_k(K;\mathcal{C})$. Define π_M, π_L the same way, such that $q \circ \pi_M = \pi_L$. Then

$$\begin{aligned}
\partial^{K,L} \circ q \circ \pi_M(\tau) &= \partial^{K,L} \circ \pi_L(\tau) \\
&= \pi_L \circ \partial^K(\tau) \\
&= q \circ \pi_M \circ \partial^K(\tau) \\
&= q \circ \partial^{K,M} \circ \pi_M(\tau)
\end{aligned}$$

Thus $\partial^{K,L} \circ q = q \circ \partial^{K,M}$ and q is a chain map. □

Again, when $\mathcal{C}$ is the constant sheaf $\mathbb{F}_K$ and M is empty, it is straightforward to conclude that we produce the usual homology short exact sequence for the pair (K, L), as the chain groups and boundary operators are equal, and the subsequent machinery is the exact same;

$$0 \longrightarrow \boldsymbol{C}_\bullet(L;\mathbb{F}) \xrightarrow{p_\bullet} \boldsymbol{C}_\bullet(K;\mathbb{F}) \xrightarrow{q_\bullet} \boldsymbol{C}_\bullet(K, L;\mathbb{F}) \longrightarrow 0$$

3 Discrete Morse Theory

In this section we heavily draw from the work of [Nan] and [CGN15]. We refer the reader to [Nan] for the definitions of partial matching, gradient path, acyclicity, critical elements, Morse chain complex.

3.1 Compatibility with Filtration and Cosheaves

Task 3. For ease of notation write $M \subset L \subset K$ as $F_1K \subset F_2K \subset F_3K$

Definition 6. *Σ is compatible with the filtration $F_\bullet K$ whenever*

$$\forall k, (\sigma \lhd \tau) \in \Sigma,\ \sigma \in F_kK \ \Leftrightarrow \tau \in F_kK$$

As we will see, this is a fundamental requirement for our incoming work. Indeed, this entails inclusion of associated Morse chain complexes.

Moreover we have one additional constraint to be satisfied in the context of cosheaves.

Definition 7. *Σ is compatible with the cosheaf $\mathcal{C}$ if*

$$\forall (\sigma \lhd \tau) \in \Sigma,\ \mathcal{C}(\tau \geq \sigma) \textit{ is invertible}.$$

We will now see how these compatibilities come into play in (relative) cosheaf homology with discrete Morse theory.

3.2 Discrete Morse Theory for Simplicial Cosheaf Homology

Before coming back to relative homology, let us focus on our regular simplicial cosheaf homology. Computing homology classes and boundary maps from our chain complex is a rather hard problem. As one could imagine, in a variety of problems, we don't expect to see a topological object appear in its minimal simplicial representation, but rather as a mess of too many simplicies in the purpose of computing homology and related objects. Indeed this is unfortunate since the linear algebra will involve huge dimensions. As explained in [CGN15], which we quote, discrete Morse theory begins with the structure of a partial matching on simplicial complex. A Morse chain complex may be constructed from this data: its chain groups are freely generated by the critical simplicies (unmatched simplicies) and the boundary operators may be derived from gradient paths. The fundamental result is that the Morse chain complex so obtained is homologically equivalent to the original simplicial complex.

Task 4. Generalization to the setting of cosheaf necessitates some additional definitions. Let $\mathcal{C}$ be a cosheaf on a simplicial complex K. Let Σ be an acyclic partial matching, compatible with the cosheaf $\mathcal{C}$. Write

$$\sigma \lhd \tau \text{ if } \sigma < \tau \text{ and } \dim(\sigma) + 1 = \dim(\tau).$$

Let $\gamma = \sigma_1 \lhd \tau_1 \rhd \cdots \rhd \sigma_m \lhd \tau_m$ be a gradient path of Σ. For convenience, write $\sigma_\gamma = \sigma_1$ and $\tau_\gamma = \tau_m$ for the first and last elements.

Definition 8. *The weight of γ with respect to the cosheaf $\mathcal{C}$ is the linear map $\mathcal{C}_\gamma : \mathcal{C}(\sigma_\gamma) \to \mathcal{C}(\tau_\gamma)$ given by*

$$\mathcal{C}_\gamma = (-\mathcal{C}(\tau_m \geq \sigma_m))^{-1} \circ \cdots \circ \mathcal{C}(\tau_1 \geq \sigma_2) \circ (-\mathcal{C}(\tau_1 \geq \sigma_1))^{-1}.$$

Take $\sigma, \tau \in C_\Sigma$ critical elements of K. The path γ is said to flow from τ to σ whenever $\sigma_\gamma \lhd \tau$ and $\sigma \lhd \tau_\sigma$.

Definition 9. *Define a linear map $\mathcal{C}^\Sigma$ on the critical elements C_Σ of K, such that for $\sigma, \tau \in C_\Sigma$, $\mathcal{C}^\Sigma_{\tau \geq \sigma} : \mathcal{C}_\tau \to \mathcal{C}_\sigma$ is defined by*

$$\mathcal{C}^\Sigma_{\tau \geq \sigma} = \mathcal{C}_{\tau \geq \sigma} + \sum_\gamma \mathcal{C}_{\tau_\gamma \geq \sigma} \circ \mathcal{C}_\gamma \circ \mathcal{C}(\tau \geq \sigma_\gamma)$$

where we sum over all gradient paths flowing from τ to σ. The sum is null if no such path exists.

This naturally induces a boundary operator $\partial_\bullet^{\mathcal{C},\Sigma}$ over the critical points C_Σ of K, where the block actions are equal to $\mathcal{C}^{\Sigma}_{\tau \geq \sigma}$.

Definition 10. *The Morse data associated to Σ consists of the critical elements C_Σ of K arranged to form the following sequence*

$$\cdots \xrightarrow{\partial_3^{\mathcal{C},\Sigma}} \boldsymbol{C}_2^{\mathcal{C},\Sigma} \xrightarrow{\partial_2^{\mathcal{C},\Sigma}} \boldsymbol{C}_1^{\mathcal{C},\Sigma} \xrightarrow{\partial_1^{\mathcal{C},\Sigma}} \boldsymbol{C}_0^{\mathcal{C},\Sigma} \longrightarrow 0$$

where $\boldsymbol{C}_k^{\mathcal{C},\Sigma} = \prod_{\tau \in C_\Sigma, dim(\tau)=k} \mathcal{C}(\tau).$

We are now prepared to prove the fundamental result in this essay.

Theorem 3.1. *Let $\mathcal{C}$ be a cosheaf on a simplicial complex K. Let Σ be an acyclic partial matching, compatible with the cosheaf $\mathcal{C}$. Then the Morse data $(\boldsymbol{C}_\bullet^{\mathcal{C},\Sigma}, \partial_\bullet^{\mathcal{C},\Sigma})$ is a chain complex. Moreover, there are isomorphisms*

$$\boldsymbol{H}_k(\boldsymbol{C}_\bullet^{\mathcal{C}}, \partial_\bullet^{\mathcal{C}}) = \boldsymbol{H}_k(\boldsymbol{C}_\bullet^{\mathcal{C},\Sigma}, \partial_\bullet^{\mathcal{C},\Sigma})$$

on homology for each dimension k.

The following subsection consists of the full proof, which goes on to show explicit chain equivalence. As previously stated, if the set C_Σ is much smaller than K, then it is vastly faster to compute the homology on the chain complex $(\boldsymbol{C}_\bullet^{\mathcal{C},\Sigma}, \partial_\bullet^{\mathcal{C},\Sigma})$.

3.3 Isomorphism with the Morse Chain Complex

Let $\mathcal{C}$ be a cosheaf on a simplicial complex K. Let Σ be an acyclic partial matching, compatible with the cosheaf $\mathcal{C}$. We prove the previous theorem by an inductive argument inspired by [CGN15]. We will iteratively work by removing one Σ-pair at a time from K. The strategy consists in updating the chain groups and the boundary operators at each stage, such that we recover the stated theorem at the final step. Informally, we will:

1. (Boundary Operator) Update the chain group and boundary operator from its previous state, in a way that is well defined at each stage. Show that the final boundary operator is as expected in the Morse chain complex
2. (Chain Equivalence) Show that there is chain equivalence at each stage

Let us begin the proof.

Choose any $\sigma \lhd \tau \in \Sigma$. Denote by $K' = K \setminus \{\sigma, \tau\}$ the reduced set of simplicies. Similarly, denote by Σ' the set $\Sigma \setminus (\sigma \lhd \tau)$, which clearly remains an acyclic partial matching on K'.

3.3.1 Boundary Operator

Definition 11. *Denote by $\partial_\bullet^{\prime\mathcal{C}}$ the boundary operator induced by the acyclic partial matching $\sigma \lhd \tau$:*

$$\partial'\big|_{x,y} = \partial\big|_{x,y} - \partial\big|_{x,\tau} \circ \partial\big|_{\sigma,\tau}^{-1} \circ \partial\big|_{\sigma,y}$$

whenever $x \lhd \tau \rhd \sigma \lhd y$. We say that we update the block by inserting $\sigma \lhd \tau$. All other blocks are left unchanged.

Theorem 3.2. *Assume that the initial sequence $(\boldsymbol{C}_\bullet(K;\mathcal{C}), \partial_\bullet^{\mathcal{C}})$ is a chain complex. Then the reduced sequence $(\boldsymbol{C}_\bullet(K';\mathcal{C}), \partial_\bullet^{\mathcal{C}\prime})$ is also a chain complex.*

Proof. Take $x, z \in K'$. For convenience, we remove the implicitly recoverable subscripts of the boundary operator. Compute:

$$\begin{aligned}(\partial' \circ \partial')\big|_{x,z} &= \sum_y \partial'\big|_{x,y} \circ \partial'\big|_{y,z} \\ &= \sum_y (\partial\big|_{x,y} - \partial\big|_{x,\tau} \circ \partial\big|_{\sigma,\tau}^{-1} \circ \partial\big|_{\sigma,y}) \circ (\partial\big|_{y,z} - \partial\big|_{y,\tau} \circ \partial\big|_{\sigma,\tau}^{-1} \circ \partial\big|_{\sigma,z}).\end{aligned}$$

By linearity:

$$\begin{aligned}(\partial' \circ \partial')\big|_{x,z} = &\sum_y \left(\partial\big|_{x,y} \circ \partial\big|_{y,z}\right) \\ &- \sum_y \left(\partial\big|_{x,y} \circ \partial\big|_{y,\tau}\right) \circ \partial\big|_{\sigma,\tau}^{-1} \circ \partial\big|_{\sigma,z}) \\ &- \partial\big|_{x,\tau} \circ \partial\big|_{\sigma,\tau}^{-1} \circ \sum_y \left(\partial\big|_{\sigma,y} \circ \partial\big|_{y,z}\right) \\ &+ \partial\big|_{x,\tau} \circ \partial\big|_{\sigma,\tau}^{-1} \circ \sum_y \left(\partial\big|_{\sigma,y} \circ \partial\big|_{y,\tau}\right) \circ \partial\big|_{\sigma,\tau}^{-1} \circ \partial\big|_{\sigma,z}\end{aligned}$$

And since $\partial \circ \partial = 0$ each term in parenthesis is zero, and in the end

$$\partial' \circ \partial' = 0$$

□

Thus the sequence at hand is a chain complex irrespective of the actual step of iteration.

There are two issues to solve to continue the procedure. First, in order to continue these successive iterations, it must be made sure that the boundary map is still invertible when needed (that is, on the remaining elements of Σ'). Second, it is not clear why the claimed formula for the final boundary map should hold after these simple updates. Let us immediately solve these.

Theorem 3.3. *The updated boundary map ∂' is still invertible on the set of critical elements Σ'.*

Proof. Take $x \triangleleft y \in \Sigma'$. Suppose, for contradiction, that $\partial'\big|_{x,y} - \partial\big|_{x,y} = -\partial\big|_{x,\tau'} \circ \partial\big|_{\tau,\tau'}^{-1} \circ \partial\big|_{\tau,y} \neq 0$. Then $\partial\big|_{x,\tau'}, \partial\big|_{\tau,y} \neq 0$ which would imply the existence of a cycle $x \triangleleft \tau \triangleright \sigma \triangleleft y \triangleright x$ in Σ, contradicting acyclicity. Thus $\partial'\big|_{x,y} = \partial\big|_{x,y} = \mathcal{C}(y \geq x)$, which is invertible. □

Solving the second problem is less straightforward. The easiest way is to show that, at each stage, there is simply an equivalence of Morse data between the initial chain complex and its reduced counterpart. To this end, it is needed to introduce a variant of the gradient path weight, based on the boundary operator rather than directly on the cosheaf.

Definition 12. *Let γ be a Σ-gradient path. The weight of γ with respect to the boundary operator (at some stage of the iteration) is the linear map $w_\gamma : \mathcal{C}(\sigma_\gamma) \to \mathcal{C}(\tau_\gamma)$ given by*

$$w_\gamma = (-\partial\big|_{\sigma_m,\tau_\gamma})^{-1} \circ \cdots \circ \partial\big|_{\sigma_2,\tau_1} \circ (-\partial\big|_{\sigma_\gamma,\tau_1})^{-1}.$$

By the previous theorem, this is well-defined at any stage. We will naturally denote by w'_γ the weight for γ' a Σ'-gradient path in any corresponding subsequent stage as

$$w'_{\gamma'} = (-\partial'\big|_{\sigma_m,\tau_{\gamma'}})^{-1} \circ \cdots \circ \partial'\big|_{\sigma_2,\tau_1} \circ (-\partial'\big|_{\sigma_{\gamma'},\tau_1})^{-1}.$$

Remark that the weight of γ with respect to the initial boundary operator before any reduction step is the weight of γ with respect to the cosheaf $\mathcal{C}$.

Theorem 3.4. *The Morse data associated to Σ is the same as the Morse data associated to Σ':*

$$(\boldsymbol{C}_\bullet^\Sigma(K;\mathcal{C}), \partial_\bullet^\Sigma) = (\boldsymbol{C}_\bullet^{\Sigma'}(K';\mathcal{C}), \partial_\bullet^{'\Sigma'}).$$

In particular this proves that the iterations recover the Morse boundary operator $\partial_\bullet^\Sigma$ at the final step, as expected.

Proof. First, note that the critical elements of K and K' are the same. This implies equality of chain groups

$$\boldsymbol{C}_\bullet^\Sigma(K;\mathcal{C}) = \boldsymbol{C}_\bullet^{\Sigma'}(K';\mathcal{C}).$$

We now focus on the linear maps $\partial_\bullet^\Sigma, \partial_\bullet'^{\Sigma'}$. Take $x \lhd y$ in K. Denote by $\mathrm{Gr}(\Sigma, x, y)$ the set of Σ-gradient paths flowing from x to y. Let us prove that

$$\partial\big|_{x,y} + \sum_{\gamma\in\mathrm{Gr}(\Sigma,x,y)} \partial\big|_{x,\tau_\gamma} \circ w(\gamma) \circ \partial\big|_{\sigma_\gamma,y} = \partial'\big|_{x,y} + \sum_{\gamma\in\mathrm{Gr}(\Sigma',x,y)} \partial'\big|_{x,\tau_\gamma} \circ w'(\gamma) \circ \partial'\big|_{\sigma_\gamma,y}$$

Take $\gamma' = x_1 \lhd y_1 \rhd \cdots \lhd y_m \in \mathrm{Gr}(\Sigma', x, y)$. There are two possibilities. Either

1. $(\sigma \lhd \tau)$ can be inserted into γ' such that it forms a gradient path $\gamma \in \Sigma$:
$$\exists! j \text{ s.t } \gamma = x_1 \lhd y_1 \rhd \cdots \lhd x_j \rhd y_j \lhd \sigma \rhd \tau \lhd x_{j+1} \rhd \cdots \lhd y_m$$
By acyclicity of Σ, we know there can be at most one index j satisfying the above equation. Then both paths γ, γ' are Σ-gradient paths. Let us examine the associated terms in the above sum:
$$\partial\big|_{x,\tau_\gamma} \circ w(\gamma) \circ \partial\big|_{\sigma_\gamma,y} + \partial\big|_{x,\tau_{\gamma'}} \circ w(\gamma') \circ \partial\big|_{\sigma_{\gamma'},y}$$
Remark that $\partial\big|_{x,\tau_\gamma} = \partial'\big|_{x,\tau_\gamma}$, as otherwise we obtain a path $\gamma \rhd \sigma \lhd \tau \rhd x$ violating acyclicity of Σ. The same argument shows $\partial\big|_{x,\sigma_\gamma} = \partial'\big|_{x,\sigma_\gamma}$. As $\tau_{\gamma'} = \tau_\gamma$ and $\sigma_{\gamma'} = \sigma_\gamma$, the previous term can be rewritten
$$\partial'\big|_{x,\tau_\gamma} \circ (w(\gamma) + w(\gamma')) \circ \partial'\big|_{\sigma_\gamma,y}$$
But see how
$$\begin{aligned} w(\gamma') + w(\gamma) &= \cdots \circ (\partial\big|_{x_{j+1},y_j} - \partial\big|_{x_{j+1},\tau} \circ \partial\big|_{\sigma,\tau}^{-1} \circ \partial\big|_{\tau,y_j}) \circ \cdots \\ &= \cdots \circ \partial'\big|_{x_{j+1},y_j} \circ \cdots \\ &= w'(\gamma') \end{aligned}$$
And thus
$$\partial\big|_{x,\tau_\gamma} \circ w(\gamma) \circ \partial\big|_{\sigma_\gamma,y} + \partial\big|_{x,\tau_{\gamma'}} \circ w(\gamma') \circ \partial\big|_{\sigma_{\gamma'},y} = \partial'\big|_{x,\tau_{\gamma'}} \circ w'(\gamma') \circ \partial'\big|_{\sigma_{\gamma'},y}$$

2. There is no such index j where the removed pair might fit, and thus the Σ'-gradient path γ' is also a Σ-gradient path. This immediately shows that $w(\gamma) = w'(\gamma')$. Actually, since γ' flows from x to y, we must have $\partial\big|_{x,\tau_{\gamma'}} = \partial'\big|_{x,\tau_{\gamma'}}$ and $\partial\big|_{\sigma_{\gamma'},y} = \partial'\big|_{\sigma_{\gamma'},y}$ (there is no way to fit the removed pair between an inclusion of contiguous simplicies). Thus
$$\partial\big|_{x,\tau_{\gamma'}} \circ w(\gamma') \circ \partial\big|_{\sigma_{\gamma'},y} = \partial'\big|_{x,\tau_{\gamma'}} \circ w'(\gamma') \circ \partial'\big|_{\sigma_{\gamma'},y}$$

Let us recollect these results to prove our theorem. Write

$$\begin{aligned} \partial^\Sigma\big|_{x,y} &= \partial\big|_{x,y} + \sum_{\gamma\in\mathrm{Gr}(\Sigma,x,y)} \partial\big|_{x,\tau_\gamma} \circ w(\gamma) \circ \partial\big|_{\sigma_\gamma,y} \\ &= \partial\big|_{x,y} - \partial\big|_{x,\tau} \circ \partial\big|_{\sigma,\tau}^{-1} \circ \partial\big|_{\sigma,y} \\ &+ \sum_{\gamma'\cup(\sigma\lhd\tau)\sim\gamma\in\mathrm{Gr}(\Sigma,x,y)} \partial\big|_{x,\tau_\gamma} \circ w(\gamma) \circ \partial\big|_{\sigma_\gamma,y} + \partial\big|_{x,\tau_{\gamma'}} \circ w(\gamma') \circ \partial\big|_{\sigma_{\gamma'},y} \\ &+ \sum_{(\sigma\lhd\tau)\not\subset\gamma\in\mathrm{Gr}(\Sigma,x,y)} \partial\big|_{x,\tau_\gamma} \circ w(\gamma) \circ \partial\big|_{\sigma_\gamma,y} \\ &= \partial'\big|_{x,y} + \sum_{(\sigma\lhd\tau)\cup\gamma'\in\mathrm{Gr}(\Sigma,x\lhd y)} \partial'\big|_{x,\tau_{\gamma'}} \circ w'(\gamma') \circ \partial'\big|_{\sigma_{\gamma'},y} \\ &+ \sum_{(\sigma\lhd\tau)\cup\gamma'\notin\mathrm{Gr}(\Sigma,x,y)} \partial'\big|_{x,\tau_{\gamma'}} \circ w'(\gamma') \circ \partial'\big|_{\sigma_{\gamma'},y} \\ &= \partial'\big|_{x,y} + \sum_{\gamma'\in\mathrm{Gr}(\Sigma',x,y)} \partial'\big|_{x,\tau_{\gamma'}} \circ w'(\gamma') \circ \partial'\big|_{\sigma_{\gamma'},y} \\ &= \partial'^{,\Sigma'}\big|_{x,y} \end{aligned}$$

which ends the proof. $\square$

Corollary 3.4.1. *Directly using Theorem 3.2 proves that the Morse data*

$$(\boldsymbol{C}^{\Sigma}_{\bullet}(K;\mathcal{C}), \partial^{\Sigma}_{\bullet})$$

and all its subsequent (equal) reductions actually form a chain complex. This entails the existence of their associated Morse homology groups

$$\boldsymbol{H}^{\Sigma}_{\bullet}(K;\mathcal{C})$$

And in particular $\boldsymbol{H}_{\bullet}(C_{\Sigma};\mathcal{C}) = \boldsymbol{H}^{\Sigma}_{\bullet}(K;\mathcal{C})$.

It remains to show that homology is preserved after each reduction step, thus obtaining

$$\boldsymbol{H}_{\bullet}(K;\mathcal{C}) = \boldsymbol{H}_{\bullet}(C_{\Sigma};\mathcal{C}) \; (= \boldsymbol{H}^{\Sigma}_{\bullet}(K;\mathcal{C}))$$

3.3.2 Chain Equivalence

We will now work to create quasi-isomorphisms of chain complexes between

$$(\boldsymbol{C}_{\bullet}(K;\mathcal{C}), \partial_{\bullet}) \text{ and } (\boldsymbol{C}_{\bullet}(K';\mathcal{C}), \partial'_{\bullet})$$

Remember that at the final step of these iterations, we recover the Morse chain complex $(\boldsymbol{C}_{\bullet}(C_{\Sigma};\mathcal{C}), \partial^{\Sigma}_{\bullet}) = (\boldsymbol{C}^{\Sigma}_{\bullet}(K;\mathcal{C}), \partial^{\Sigma}_{\bullet})$ associated to K.

Define the linear map $\Psi_{\bullet} : \boldsymbol{C}_{\bullet}(K;\mathcal{C}) \to \boldsymbol{C}_{\bullet}(K';\mathcal{C})$ by the following block action

$$\Psi\big|_{\omega,\alpha} = \begin{cases} -\partial\big|_{\omega,\tau} \circ \partial^{-1}\big|_{\sigma,\tau} & \text{if } \alpha = \sigma \\ Id & \text{if } \alpha = \omega \\ 0 & \text{else} \end{cases}$$

Similarly define $\Phi_{\bullet} : \boldsymbol{C}_{\bullet}(K';\mathcal{C}) \to \boldsymbol{C}_{\bullet}(K;\mathcal{C})$ by the following block action

$$\Phi\big|_{\alpha,\omega} = \begin{cases} -\partial^{-1}\big|_{\sigma,\tau} \circ \partial\big|_{\sigma,\omega} & \text{if } \alpha = \tau \\ Id & \text{if } \alpha = \omega \\ 0 & \text{else} \end{cases}$$

Theorem 3.5. $\Psi_{\bullet}$ *is a chain map.*

Proof. We want to prove that

$$\Psi \circ \partial = \partial' \circ \Psi$$

Take $\alpha \in \boldsymbol{C}_{\bullet}(K;\mathcal{C})$ and $\omega \in \boldsymbol{C}_{\bullet}(K';\mathcal{C})$. Again we will not bother with subscripts/dimension. Let us study the block action of each side of the previous equation. The left hand side becomes

$$\Psi \circ \partial\big|_{\omega,\alpha} = \sum_{c \in K} \Psi\big|_{\omega,c} \circ \partial\big|_{c,\alpha}$$

which is non zero for $c = \sigma, \omega$. Thus

$$\Psi \circ \partial\big|_{\omega,\alpha} = \partial\big|_{\omega,\alpha} - \partial\big|_{\omega,\tau} \circ \partial^{-1}\big|_{\sigma,\tau} \circ \partial\big|_{\sigma,\alpha} = \partial'\big|_{\omega,\alpha}$$

Now the right hand side becomes

$$\partial' \circ \Psi\big|_{b,\alpha} = \sum_{c \in K'} \partial'\big|_{\omega,c} \circ \Psi\big|_{c,\alpha}$$

which trivially equals $\partial'\big|_{\omega,\alpha}$ whenever $\alpha \neq \sigma$. When $\alpha = \sigma$, the right hand side becomes:

$$\sum_{c \in K'} \partial'\big|_{\omega,c} \circ \Psi\big|_{c,\alpha} = -\sum_{c \in K'} \partial\big|_{\omega,c} \circ \partial\big|_{c,\tau} \circ \partial^{-1}\big|_{\sigma,\tau} + \sum_{c \in K'} \partial\big|_{\omega,\tau} \circ \partial^{-1}\big|_{\sigma,\tau} \circ \partial\big|_{\sigma,c} \circ \partial\big|_{c,\tau} \circ \partial^{-1}\big|_{\sigma,\tau}$$

Since $\dim(c) + 1 \neq \dim(\sigma)$ or $\dim(c) + 1 \neq \dim(\tau)$, the second sum is zero. Moreover, remember that ∂ is a valid boundary operator, thus $\partial \circ \partial = 0$. This shows that

$$\begin{aligned}\partial' \circ \Psi\big|_{\omega,\alpha} &= -\sum_{c \in K'} \left(\partial\big|_{\omega,c} \circ \partial\big|_{c,\tau}\right) \circ \partial^{-1}\big|_{\sigma,\tau} \\ &= \partial\big|_{\omega,\sigma} \circ \partial\big|_{\sigma,\tau} \circ \partial^{-1}\big|_{\sigma,\tau} \\ &= \partial\big|_{\omega,\sigma} = \partial\big|_{\omega,\alpha}\end{aligned}$$

and this concludes the proof. □

The same arguments can be used for the other map Φ.

Theorem 3.6. $\Phi_\bullet$ *is a chain map.*

Proof. Use the same exact method and arguments as for $\Psi_\bullet$. □

It is straightforward to verify that $\Psi_\bullet \circ \Phi_\bullet$ is the identity map on $\boldsymbol{C}_\bullet(K'; \mathcal{C})$. It remains to construct a chain homotopy between $\Phi_\bullet \circ \Psi_\bullet$ and the identity on $\boldsymbol{C}_\bullet(K; \mathcal{C})$.

Theorem 3.7. *The linear maps* $\Theta_n : \boldsymbol{C}_n(K; \mathcal{C}) \to \boldsymbol{C}_{n+1}(K; \mathcal{C})$ *defined by the block action* $\mathcal{C}(\alpha) \to \mathcal{C}(\beta)$

$$\Theta_{\beta,\alpha} = \begin{cases} \partial^{-1}\big|_{\sigma,\tau} & \textit{if } \alpha = \sigma, \beta = \tau \\ 0 & \textit{else} \end{cases}$$

constitute a chain homotopy between $\Psi \circ \Phi$ *and* Id *on* $\boldsymbol{C}_\bullet(K; \mathcal{C})$. *This results completes our proof of Theorem 3.1*

Proof. It is straightforward to verify that

$$(Id - (\Theta \circ \partial + \partial \circ \Theta)) = (\Psi \circ \Phi)\,.$$

By definition of chain homotopy, this suffices to end the proof. □

4 Discrete Morse Theory for Relative Cosheaf Homology

4.1 Morse Relative Cosheaf Homology and its Long Exact Sequence

Let us come back to our original concern. All the work done until now will let us easily define and justify a generalization of discrete Morse theory to relative homology. First, notice that requiring Σ to be adapted to our filtration implies inclusion of their respective Morse chain complex. Again, for simplicity, denote $M \subset L \subset K$ by $F_1K \subset F_2K \subset F_3K$

Theorem 4.1. *Assume that Σ is adapted to the filtration (F_kK) of simplicial complexes. Then there is an inclusion of their respective Morse chain complexes*

$$\boldsymbol{C}_\bullet^\Sigma(F_kK;\mathcal{C}) \hookrightarrow \boldsymbol{C}_\bullet^\Sigma(F_{k+1}K;\mathcal{C})$$

Proof. First, as the critical simplicies of F_kK are a subset of the critical simplicies of $F_{k+1}K$, the inclusion of chain groups is obviously well defined. Now it remains to show that the inclusion map commutes with the Morse boundary operator. This is equivalent to the following statement

$$\partial_\bullet^{\Sigma,F_{k+1}K}\big|_{\boldsymbol{C}_\bullet^\Sigma(F_kK;\mathcal{C})} = \partial_\bullet^{\Sigma F_kK}$$

Recall that the definition of the Morse boundary operator only involves Σ-gradient paths that flow from source simplex x to target simplex y:

$$\partial_\bullet^{\Sigma,F_{k+1}K}\big|_{x,y} = \partial_\bullet^{\Sigma,F_kK}\big|_{x,y} + \sum_\gamma \partial_\bullet^{\Sigma,F_kK}\big|_{x,\tau_\gamma} \circ w_\gamma^{F_kK} \circ \partial_\bullet^{\Sigma,F_kK}\big|_{\sigma\gamma,y}$$

Since Σ is adapted to the filtration at hand, we know that any gradient path γ flowing between elements of $x, y \in F_kK$ must pass through elements of F_kK. Thus $w^{F_{k+1}K}(\gamma)$ only contains boundary maps between elements of F_kK. As described in the very beginning of this essay, the boundary maps commute with inclusion, that is

$$\partial_\bullet^{\Sigma,F_{k+1}K}\big|_{F_kK} = \partial_\bullet^{\Sigma,F_kK}.$$

which very clearly proves the desired result.

□

This property of inclusion of Morse chain complex leads us to the following quotient chain complex.

Definition 13. *Define the relative Morse chain complex to be*

$$\left(\boldsymbol{C}_\bullet^\Sigma(F_{k+1}K, F_kK;\mathcal{C}), \partial_\bullet^{\Sigma,F_{k+1}K,F_kK}\right)$$

where $\boldsymbol{C}_\bullet^\Sigma(F_{k+1}K, F_kK;\mathcal{C}) = \dfrac{\boldsymbol{C}_\bullet^\Sigma(F_{k+1}K;\mathcal{C})}{\boldsymbol{C}_\bullet^\Sigma(F_kK;\mathcal{C})}$ *and* $\partial_\bullet^{\Sigma,F_{k+1}K,F_kK}$ *is its induced boundary operator, which is well-defined by the previous theorem.*

Definition 14. *Define the inclusion $p_\bullet^\Sigma$ and projection $q_\bullet^\Sigma$ operators in the exact same way as we did for relative cosheaf homology. This makes the following sequence exact:*

$$0 \longrightarrow \boldsymbol{C}_\bullet^\Sigma(L,M;\mathcal{C}) \xrightarrow{p_\bullet^\Sigma} \boldsymbol{C}_\bullet^\Sigma(K,M;\mathcal{C}) \xrightarrow{q_\bullet^\Sigma} \boldsymbol{C}_\bullet^\Sigma(K,L;\mathcal{C}) \longrightarrow 0$$

Theorem 4.2. *The inclusion and projection maps $p_\bullet^\Sigma, q_\bullet^\Sigma$ are chain maps.*

Proof. Same exact proof as in 2.3, since we only use properties of quotient spaces and their associated maps. □

We are almost at the end of our construction. We would like to connect the relative cosheaf homology groups of the initial chain complexes to their Morse counterpart. To do this we need to construct an ultimate object; the quasi-isomorphism between the two chain complexes.

Recall how we built our quasi-isomorphisms Ψ^{F_kK} between the original and the Morse chain complex. Consider some $k' < k$. Consider a stage of the iterations where we have selected some $\sigma \lhd \tau \in F_kK \setminus F_{k'}K$. When Σ is adapted to the filtration at hand, the corresponding $\Psi^{F_kK}_{\sigma,\bullet}$ is the identity map on the subspace corresponding to the simplicies of $F_{k'}K$. Thus, the restriction of $\Psi^{F_kK}_{\bullet}$ to $F_{k'}K$ is equal to the quasi-isomorphism that would have been built between the original chain complex and its Morse counterpart associated to $F_{k'K}$:

$$\Psi^{F_kK}_{\bullet}\big|_{F_{k'K}} = \Psi^{F_{k'}K}_{\bullet}$$

We could also say that $\Psi_\bullet$ commutes with inclusion. This justifies the following definition.

Definition 15. *Define $\Psi^{F_kK,F_{k'}K}$ to be the induced quasi-isomorphism on the quotient Morse chain complex:*

$$\Psi^{F_kK,F_{k'}K} : \boldsymbol{C}_\bullet(F_kK, F_{k'}K;\mathcal{C}) \longrightarrow \boldsymbol{C}^\Sigma_\bullet(F_kK, F_{k'}K;\mathcal{C})$$

It is possible to prove these are indeed quasi-isomoprhism by the same arguments used before, this time in quotient space.

This immediately leads to the following.

Theorem 4.3. *The induced quasi-isomorphisms $\Psi^{F_kK,F_{k'}K}$, $k > k'$, are such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{C}_\bullet(L,M;\mathcal{C}) & \xrightarrow{p_\bullet} & \boldsymbol{C}_\bullet(K,M;\mathcal{C}) & \xrightarrow{q_\bullet} & \boldsymbol{C}_\bullet(K,L;\mathcal{C}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Psi^{L,M}_\bullet} & & \downarrow{\scriptstyle \Psi^{K,M}_\bullet} & & \downarrow{\scriptstyle \Psi^{K,L}_\bullet} & & \\
0 & \longrightarrow & \boldsymbol{C}^\Sigma_\bullet(L,M;\mathcal{C}) & \xrightarrow{p^\Sigma_\bullet} & \boldsymbol{C}^\Sigma_\bullet(K,M;\mathcal{C}) & \xrightarrow{q^\Sigma_\bullet} & \boldsymbol{C}^\Sigma_\bullet(K,L;\mathcal{C}) & \longrightarrow & 0
\end{array}$$

Proof. Same exact proof as in Theorem 2.3, since we only use properties of quotient spaces and their associated maps (here, working with Ψ instead of the boundary operator). □

Now we use the admitted result in the Mini-project guidance, which is called the naturality of the connecting homomorphism. Let $S_\bullet$ and $S^\Sigma_\bullet$ be the connecting homomorphism of our initial relative cosheaf chain complexes and their Morse counterpart, in the long exact sequence given by the snake lemma. Our maps $\Psi^{F_kK,F_{k'}K}$, being quasi-isomorphism, they induce an isomorphism between the following long exact sequence of homology groups:

$$\begin{array}{ccccccc}
\cdots \boldsymbol{H}_d(L,M;\mathcal{C}) & \xrightarrow{H_p} & \boldsymbol{H}_d(K,M;\mathcal{C}) & \xrightarrow{H_q} & \boldsymbol{H}_d(K,L;\mathcal{C}) & \xrightarrow{S_d} & \boldsymbol{H}_{d-1}(L,M;\mathcal{C}) \cdots \\
\downarrow{\scriptstyle H_{\Psi^{L,M}_d}} & & \downarrow{\scriptstyle H_{\Psi^{K,M}_d}} & & \downarrow{\scriptstyle H_{\Psi^{K,L}_d}} & & \downarrow{\scriptstyle H_{\Psi^{L,M}_{d-1}}} \\
\cdots \boldsymbol{H}^\Sigma_d(L,M;\mathcal{C}) & \xrightarrow{H_{p^\Sigma}} & \boldsymbol{H}^\Sigma_d(K,M;\mathcal{C}) & \xrightarrow{H_{q^\Sigma}} & \boldsymbol{H}^\Sigma_d(K,L;\mathcal{C}) & \xrightarrow{S^\Sigma_d} & \boldsymbol{H}^\Sigma_{d-1}(L,M;\mathcal{C}) \cdots
\end{array}$$

where of course $\boldsymbol{H}^\Sigma_\bullet = \boldsymbol{H}_\bullet$. In the end, this construction lets us greatly accelerate the computation of the connecting homomorphisms $S_d : \boldsymbol{H}_d(K,L;\mathcal{C}) \to \boldsymbol{H}_{d-1}(L,M;\mathcal{C})$, since we now have

$$S_d = (H_{d-1}\Psi^{L,M})^{-1} \circ S^\Sigma_d \circ H_d\Psi^{K,L}.$$

4.2 Computation Improvements

Remember that the construction of the homomorphisms S_d, S^Σ_d consists in linear algebra as described in the construction of the zig-zag lemma. Algorithmic complexity for the operations needed, say computing matrix inverse or matrix null space (Gauss-Jordan elimination, SVD), scale like

$$O(n^3),$$

where n is the number of simplicies in K. When working with the Morse chain complex $\boldsymbol{C}^\Sigma_\bullet$, the effective number of simplicies used in the linear algebra is n' the number of critical simplicies, which can be much lower. Let us give a slightly more precise complexity analysis. [All+19] describe an efficient algorithm to compute an acyclic partial matching adapted to some filtration. The complexity analysis (see section 3.3) involves sorting faces of all the elements of K; the complexity of the worst case scenario is upper bounded by

$$O(n^2 \log(n)).$$

Then, [CGN15] describes an algorithm, `Scythe`, to compute the Morse chain complex from Σ an acyclic partial matching on K, adapted to a sheaf $\mathcal{F}$. The work could be generalized to cosheaves. The time complexity of the algorithm is $O(np\tilde{m}d^3)$ where

1. $p = \max_{x \in X} |\{y \in K \mid x \triangleleft y\}|$
2. $d = \max_{x \in X} \text{rank}\mathcal{C}(x)$
3. m_k is the number of critical elements of dimension k
4. $\tilde{m} = \sum_k m_k^2 = O(n'^2)$

Thus the time complexity of `Scythe` is upper bounded by

$$O(nn'^2 pd^3).$$

Then, the subsequent linear algebra is of much lower $O(n'^3)$ complexity. In total, this is better than the initial $O(n^3)$ complexity; this explicitly shows the expected gains in computation time. One can refer to the cited articles to make sure that the space complexity is also lower than naive computations.

5 Generalisation to Longer Filtrations: Persistent Cosheaf Homology

Task 5. Let us discuss the possible generalization when considering a larger filtration

$$F_1K \subset F_2K \subset \cdots \subset \cdots F_nK = K$$

The only reasonable direction that comes to mind is constructing persistent cosheaf homology, similarly to the work of [Nan]. Let us adapt the machinery of discrete Morse theory for persistent homology computation in the context of cosheaves. This will be a rather straightforward generalization, considering all that has been done until now. We require the same conditions for our acyclic partial matching Σ to be compatible with our filtration (see Definition 6). Theorem 4.1 shows there is therefore an inclusion of Morse chain complexes $\boldsymbol{C}^{\Sigma}_{\bullet}(F_kK;\mathcal{C}) \xhookrightarrow{i^k_\bullet} \boldsymbol{C}^{\Sigma}_{\bullet}(F_{k+1}K;\mathcal{C})$. We now use similar arguments as in Theorem 8.15 of [Nan], proof of which we practically quote here.

Theorem 5.1. *For any pair $i < j$ and dimension $k \geq 0$, there are isomoprhisms*

$$\boldsymbol{PH}_{i\to j}\boldsymbol{H}_k(F_\bullet K;\mathcal{C}) \sim \boldsymbol{PH}_{i\to j}\boldsymbol{H}^{\Sigma}_k(F_\bullet K;\mathcal{C})$$

of persistent homology groups. The barcodes of $\boldsymbol{H}_k(F_\bullet K;\mathcal{C})$ and $\boldsymbol{H}^{\Sigma}_k(F_\bullet K;\mathcal{C})$ are thus equal.

Proof. For any pair $i < j$ and dimension $k \geq 0$, the following diagrams of vector spaces commute

$$\begin{array}{ccc} \boldsymbol{C}_k(F_kK;\mathcal{C}) & \hookrightarrow & \boldsymbol{C}_k(F_{k+1}K;\mathcal{C}) \\ \downarrow \Psi^{F_kK}_k & & \downarrow \Psi^{F_{k+1}K}_k \\ \boldsymbol{C}^{\Sigma}_k(F_kK;\mathcal{C}) & \hookrightarrow & \boldsymbol{C}^{\Sigma}_k(F_{k+1}K;\mathcal{C}) \end{array} \qquad \begin{array}{ccc} \boldsymbol{C}_k(F_kK;\mathcal{C}) & \hookrightarrow & \boldsymbol{C}_k(F_{k+1}K;\mathcal{C}) \\ \Phi^{F_kK}_k \uparrow & & \Phi^{F_{k+1}K}_k \uparrow \\ \boldsymbol{C}^{\Sigma}_k(F_kK;\mathcal{C}) & \hookrightarrow & \boldsymbol{C}^{\Sigma}_k(F_{k+1}K;\mathcal{C}) \end{array}$$

Since Ψ, Φ form two halves of a chain homotopy equivalence, they induce isomoprhisms on k-th homology for all $k \geq 0$. Thus, we obtain a 0-interleaving between the two k-th homology persistence modules, which guarantees isomorphisms of their persistent homology groups. □

In the same fashion, this shows how computation of persistent cosheaf homology groups, along with the inclusion maps corresponding to the corresponding filtration, can be considerably sped up thanks to discrete Morse theory.

In conclusion, we have described a pretty comprehensive method to adapt relative and persistent cosheaf homology to discrete Morse theory. Possible research that remain, in the continuity of this work, can be in the direction of algorithms for building adapted acyclic partial matching. This can consist of obtaining the best possible matching, eliminating a maximum of simplicies, or in obtaining the fastest algorithms to produce interesting matchings.

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